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Journal of Pure and Applied Algebra 194 (2004) 127–145

JOURNAL OF PURE AND APPLIED ALGEBRA

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Jordan systems of Martindale-like quotients

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Received 14 July 2003; received in revised form 16 February 2004

Communicated by C.A. Weibel

Abstract

In this paper we introduce the notion of Jordan system (algebra, pair or triple system) of Martindale-like quotients with respect to a filter of ideals as that whose elements are absorbed into the original system by ideals of the filter, and prove that it inherits regularity conditions such as (semi)primeness and nondegeneracy. When we consider power filters of sturdy ideals, the notions of Jordan systems of Martindale-like quotients and Lie algebras of quotients are related through the Tits–Kantor–Koecher construction, and that allows us to give constructions of the maximal systems of quotients when the original systems are nondegenerate.

MSC: 17C10; 17C50; 17S70

The theory of rings of quotients has its origins between 1930 and 1940, in the works of Ore and Osano on the construction of the total ring of fractions. In that decade Ore proved that a necessary and sufficient condition for a ring R to have a (left) classic ring of quotients is that for any regular element a in R, and any $b \in R$ there exist a regular $c \in R$ and $d \in R$ such that $cb = da$ (left Ore condition). At the end of 1950s, Goldie, Lesieur and Goisot characterized the (associative) rings that are classic left orders in semiprime and left artinian rings [19, Chapter IV] (result known as Goldie's Theorem).

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1 Partially supported by the MCYT and Fondos FEDER, BFM2001-1938-C02-02.

2 Partially supported by the MCYT and Fondos FEDER, BFM2001-1938-C02-01, and the Junta de Andalucía FQM264.

0022-4049/$ - see front matter

doi:10.1016/j.jpaa.2004.04.001

Later on in 1956, Utumi introduced the notion of general ring of quotients [34] and proved that the rings without right zero divisors are precisely those which have left maximal rings of quotients.

Martindale rings of quotients were introduced by Martindale in 1969 for prime rings [21]. This concept was designed for applications to rings satisfying a generalized polynomial identity (GPI for short). In his work, Martindale showed that every prime ring satisfying a GPI is a subring of a primitive ring Q with nonzero socle and, moreover, the division ring associated to the socle of Q is finite dimensional over its center (this result generalizes both Amistur's and Posner's theorems).

In 1972, Amitsur generalized the construction of Martindale rings of quotients to the setting of semiprime rings; see [1]. This notion has proven to be useful not only for the theory of rings with identities, but also for Galois theory on noncommutative rings and for the study of prime ideals under ring extensions in general.

In 1989, McCrimmon, when studying Jordan algebras and triple systems of symmetric elements, generalized Martindale's quotients (in the setting of associative and not necessarily semiprime rings) introducing the notion of Martindale rings of quotients relative to a filter of "denominators"; see [26].

It is natural to ask whether similar notions (and results) can be obtained for Jordan algebras. The question of Goldie's Theorems for Jordan algebras was posed by Jacobson [12] and was studied in the case of the special Jordan algebras $J = H(A,*)$ by Britten and Montgomery; see [4–6,30]. A definitive answer was given by Zelmanov in [37,38] making use of his fundamental result on structure theory of strongly prime Jordan algebras. In [8], Fernández López et al. showed that Zelmanov's version of Goldie's conditions still characterize quadratic algebras having an artinian algebra of quotients which is simple or nondegenerate, according to whether the original algebra is strongly prime or nondegenerate.

In a recent paper (see [22]), Martínez gave an Ore-type condition for a Jordan algebra to have a classical algebra of fractions. Moreover, making use of the Tits–Kantor–Koecher construction that relates Jordan and Lie structures, she built a maximal Jordan algebra of quotients as a direct limit of derivations defined on certain inner ideals. This idea of considering equivalence classes of derivations defined on ideals is the key point of Siles Molina construction of maximal Lie algebras of quotients [32].

In this paper, we introduce the notion of Martindale-like systems of quotients of a Jordan system J with respect to a filter as those whose elements are absorbed into J by ideals of the filter. We have chosen the name "Martindale-like system of quotients" since they behave like Martindale rings of quotients in the sense that the absorption property involves ideals (and not inner ideals as in [22]).

Moreover, inspired by Martínez' idea of moving from a Jordan setting to a Lie one through the Tits–Kantor–Koecher construction and using the construction of maximal Lie algebras of quotients of Siles Molina [32], we give explicit constructions of maximal Jordan systems of Martindale-like quotients for nondegenerate Jordan pairs, triple systems and algebras with respect to power filters of ideals. In order to recover a Jordan system from the maximal Lie algebra of quotients of the Tits–Kantor–Koecher algebra of a Jordan system, we will use the extra hypothesis of having $\frac{1}{6}$ in the ring of scalars, so we have decided to work in the "linear Jordan setting" even for the definitions and first properties of Martindale-like systems of quotients.

0. Preliminaries

0.1. We will deal with Lie algebras and with Jordan systems over a ring of scalars Φ with $\frac{1}{2}$. The reader is referred to [11,12,20,27] for basic results, notation and terminology, though we will stress some notions and basic properties.

- Given a Lie algebra L, its product will be denoted by $[x, y]$, for $x, y \in L$. It satisfies the Jacobi identity and $[x,x] = 0$ for any $x \in L$.
- For a Jordan pair $V = (V^+, V^-)$ we will denote the products by $Q_x y$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$, with linearizations denoted by $Q_{x,z} y = \{x, y, z\} = D_{x,y} z$.
- A Jordan triple system T is given by its products $P_x y$, for any $x, y \in T$, with linearizations denoted by $P_{x,z} y = \{x, y, z\} = L_{x,y} z$.
- Given a Jordan algebra J, its products will be denoted x^2, $U_x y$, for $x, y \in J$. They are quadratic in x and linear in y and have linearizations denoted $x \circ y$, $U_{x,z} y = \{x, y, z\} = V_{x,y} z$, respectively. Moreover, from [23, 0.4], for any $x, y, z \in J$

$$(x \circ y) \circ z = \{x, y, z\} + \{y, x, z\}. \tag{1}$$

If there exists an element $1 \in J$ such that $U_1 x = x$ and $U_x 1 = x^2$ for every $x \in J$, we will say that J is a *unital* Jordan algebra.

In any case, since we will work under the assumption of $\frac{1}{2} \in \Phi$, we will consider mainly linear products in all Jordan structures because they completely determine the quadratic Jordan products of the system. In particular, in the algebra case, it suffices to consider the linearization of the square $\circ$ because it is related with the linear triple product by

$$2\{x, y, z\} = (x \circ y) \circ z - (x \circ z) \circ y + (y \circ z) \circ x \tag{2}$$

(cf. [24, 1.4] linearized on the first variable).

0.2. A Jordan algebra J gives rise to a Jordan triple system J_T by simply forgetting the squaring and letting $P = U$. Moreover, J is nondegenerate if and only if J_T is so. Conversely, if a Jordan triple system T has an element 1 with $P_1 x = x$ for every $x \in T$, then it is really a unital Jordan algebra with product $U = P$ and $x^2 = P_x 1$ [28, 0.1].

By doubling any Jordan triple system T one obtains the *double Jordan pair* $V(T) = (T, T)$ with products $Q_x y = P_x y$, for any $x, y \in T$ [20, 1.13]. Moreover, T is a nondegenerate Jordan triple system if and only if $V(T)$ is nondegenerate.

0.3. We recall the notions of annihilators in Jordan systems when $\frac{1}{2} \in \Phi$ [25, 1.2]:

- Let V be a Jordan pair and let $X = (X^+, X^-)$ be a subset of V. Then the *annihilator of X in V* is $\mathrm{Ann}_V(X) = (\mathrm{Ann}_V(X)^+, \mathrm{Ann}_V(X)^-)$ where, for $\sigma = \pm$,

$$\mathrm{Ann}_V(X)^\sigma = \{z \in V^\sigma \mid \{z, X^{-\sigma}, V^\sigma\} = \{z, V^{-\sigma}, X^\sigma\} = \{V^{-\sigma}, z, X^{-\sigma}\} = 0\}.$$

- Let T be a Jordan triple system and X be a subset of T. The *annihilator of X in T* is

$$\mathrm{Ann}_T(X) = \{z \in T \mid \{z, T, X\} = \{z, X, T\} = \{X, z, T\} = 0\}.$$

- Let J be a Jordan algebra and let X be a subset of J. The *annihilator of X in J* is defined as

$$\mathrm{Ann}_J(X) = \{z \in J \mid z \circ X = \{z, X, J\} = 0\}.$$

It is not hard to prove that this definition coincides with the one given in [25] where it is shown that the annihilator of an ideal in a Jordan system is always an ideal. An ideal of a Jordan system J is called *sturdy* if it has zero annihilator in J.

0.4. Lemma. *If J is a Jordan algebra and I is an ideal of J, then $\mathrm{Ann}_J(I) \subseteq \mathrm{Ann}_{J_T}(I)$. Moreover, the equality holds when J has zero annihilator.*

Proof. By (0.1)(2) it is clear that $\mathrm{Ann}_J(I) \subseteq \mathrm{Ann}_{J_T}(I)$. Moreover, if $\mathrm{Ann}_J(J) = 0$ and $z \in \mathrm{Ann}_{J_T}(I)$, then $0 = \{z, y, x\} + \{y, z, x\} = (z \circ y) \circ x$ for every $x \in J$, $y \in I$ by (0.1)(1), so $z \circ I \in \mathrm{Ann}_J(J) = 0$, proving that $z \in \mathrm{Ann}_J(I)$. □

1. Derivations in 3-graded Lie algebras

1.1. A $(2n+1)$-*grading* of a Lie algebra L is a decomposition $L = \bigoplus_{i=-n}^{n} L_i$ where each L_i is a submodule of L, satisfying $[L_i, L_j] \subseteq L_{i+j}$, where $L_{i+j} = 0$ if $|i+j| > n$. In this paper we will focus mainly on 3-graded Lie algebras.

1.2. Given a 3-graded Lie algebra L, the formula $\{x, y, z\} = [[x, y], z]$ defines a Jordan pair structure on (L_1, L_{-1}) as soon as $\frac{1}{6} \in \Phi$ (cf. [20, 2.2(b); 32, 1.2]).

Conversely, from a Jordan pair $V = (V^+, V^-)$ we can always build a 3-graded Lie algebra through the Tits–Kantor–Koecher (TKK for short) construction [31, Section 1]. This type of Lie algebras was first considered by Tits [33], Kantor [14–16] and Koecher [17,18], which justifies the notation $\mathrm{TKK}(V)$.

1.3. Let L be a Lie algebra and I an ideal of L. Then for every isomorphism of Lie algebras $g: L \to L$, and every derivation with domain I, $d: I \to L$, $g^{-1}dg$ is a derivation of L defined on $g^{-1}(I)$ since $g^{-1}dg([a,b]) = g^{-1}d([g(a), g(b)]) = g^{-1}([dg(a), g(b)] + [g(a), dg(b)]) = [g^{-1}dg(a), b] + [a, g^{-1}dg(b)]$ for all $a, b \in g^{-1}(I)$.

1.4. If $f: V \to W$ is an isomorphism of Jordan pairs, then $\mathrm{TKK}(f)$: $\mathrm{TKK}(V) \to \mathrm{TKK}(W)$ defined by $\mathrm{TKK}(f)(x^+ + d + y^-) = f^+(x^+) + fdf^{-1} + f^-(y^-)$ is an isomorphism of Lie algebras. Conversely, any isomorphism $\mathrm{TKK}(V) \to \mathrm{TKK}(W)$ respecting the 3-gradings arises in this way [10, 2.2(6)].

1.5. In particular, for a Jordan triple system T the exchange isomorphism $\mathrm{ex}: V(T) \to V(T)$ defined as $\mathrm{ex}((x, y)) = (y, x)$ for any $(x, y) \in V(T)$ can be extended to a Lie algebra isomorphism $\mathrm{TKK}(\mathrm{ex})$ of the Lie algebra $\mathrm{TKK}(V(T))$ by (1.4). Therefore, for every ideal I of $\mathrm{TKK}(V(T))$ and every derivation $d \in \mathrm{Der}(I, \mathrm{TKK}(V(T)))$, we get another derivation

$$d^{\mathrm{ex}} = \mathrm{TKK}(\mathrm{ex})^{-1} d\, \mathrm{TKK}(\mathrm{ex}) = \mathrm{TKK}(\mathrm{ex}) d\, \mathrm{TKK}(\mathrm{ex})$$

defined on $\mathrm{TKK}(\mathrm{ex})(I)$ (cf. (1.3)).

1.6. Given a 3-graded ideal $I = I_1 \oplus I_0 \oplus I_{-1}$ of a 3-graded Lie algebra $L = L_1 \oplus L_0 \oplus L_{-1}$ and a derivation $d : I \to L$, we say that d has *degree* i if $d(I_j) \subseteq L_{j+i}$, for all $j = 0, \pm 1$.

1.7. Proposition. *Let I be a nondegenerate 3-graded ideal of a 6-torsion free Lie algebra L and let $d : I \to L$ be a derivation on I of degree 2 or -2. Then d is zero on $\tilde{I}_d = \{x \in I \mid d(x) \in I\}$, which is a 3-graded ideal of I containing $[I, I]$.*

Proof. First notice that $\tilde{I}_d$ is a 3-graded ideal of I and that it is nondegenerate because I is [36, Lemma 4].

Suppose that d is a derivation of degree 2 on I. Since $d^2 = 0$ on $\tilde{I}_d$ and L has no 2-torsion, for any $a \in \tilde{I}_d$, $0 = \mathrm{ad}(d^2(a)) = [d, \mathrm{ad}(d(a))] = [d, [d, \mathrm{ad}\, a]] = -2d(\mathrm{ad}\, a)d$ on $\tilde{I}_d$, which implies

$$d(\mathrm{ad}\, a)d = 0 \quad \text{on } \tilde{I}_d. \tag{3}$$

Moreover, if for any a, $b \in \tilde{I}_d$ we denote $A := \mathrm{ad}\, a$ and $B := \mathrm{ad}\, b$ and use (3),

$$0 = d[A, [A, [B, [B, d]]]]d = 4dABdBAd + dA^2dB^2d + dB^2dA^2d \quad \text{on } \tilde{I}_d. \tag{4}$$

Putting $a = b$ in formula (4), we obtain $6dA^2dA^2d = 0$ on $\tilde{I}_d$, so L being 6-torsion free implies

$$dA^2dA^2d = 0 \quad \text{on } \tilde{I}_d. \tag{5}$$

Now assume that $a \in (\tilde{I}_d)_{-1}$. In this case, it is clear that $A^3 = 0$, hence

$$[[d, A], [[d, A], A]] = [(dA - Ad), (dA^2 + A^2d - 2AdA)] = -3dA^2dA - 3AdA^2d$$

on $\tilde{I}_d$, which implies

$$[[d, A], [[d, A], A]]^2 = 9dA^2dA^2dA^2d = 0 \quad \text{on } \tilde{I}_d,$$

taking into account (3) and (5). We have just shown that the inner derivation $[[d, A], [[d, A], A]] = \mathrm{ad}([d(a), [d(a), a]])$ has zero square in $\tilde{I}_d$, so by nondegeneracy of $\tilde{I}_d$ it is zero itself, giving $-3dA^2dA - 3AdA^2d = 0$ on $\tilde{I}_d$, hence

$$dA^2dA = -AdA^2d \quad \text{on } \tilde{I}_d. \tag{6}$$

Moreover, multiplying (4) on the right by A^2d and using (3) and (6) we get,

$$\begin{aligned} 0 &= 4dABdBAdA^2d + dA^2dB^2dA^2d + dB^2dA^2dA^2d \\ &= -4dABdBdA^2dA + dA^2dB^2dA^2d - dB^2dAdA^2dA \\ &= dA^2dB^2dA^2d \text{ on } \tilde{I}_d. \end{aligned} \tag{7}$$

Finally, for any $b \in \tilde{I}_d$ and $a \in (\tilde{I}_d)_{-1}$

$$\begin{aligned}[][[d,A],[[d,A],B]] &= [(dA - Ad),(dAB + BAd - AdB - BdA)] \\ &= dABAd - dA^2dB - dABdA - AdBAd - dABdA \\ &\quad + dABAd - AdBAd - BdA^2d\end{aligned}$$

on $\tilde{I}_d$, hence its square is

$$\begin{aligned}[][[d,A],[[d,A],B]]^2 &= 2dA^2dBAdBAd + 4dABdA^2dBAd + 2dABdABdA^2d \\ &= 2dA^2dABdBAd + 4dABdA^2dABd + 2dABdBAdA^2d \\ &= -2AdA^2dBdBAd - 4dABAdA^2dBd - 2dABdBdA^2dA \\ &= 0 \quad \text{on } \tilde{I}_d\end{aligned}$$

by (3), (6) and (7), and taking into account that $dABd = dBAd$ since $d[A,B]d = 0$ by (3). Then the inner derivation $[[d,A],[[d,A],B]]$ has zero square on $\tilde{I}_d$, so it is zero itself, giving $[[d,A],[[d,A],B]]=0$ on $\tilde{I}_d$ for any $b \in \tilde{I}_d$. And using again nondegeneracy of $\tilde{I}_d$ we get $[d,A] = \operatorname{ad} d(a) = 0$ on $\tilde{I}_d$ for any $a \in (\tilde{I}_d)_{-1}$, so d acts trivially on the only graded part of $\tilde{I}_d$ on which it could be nonzero, i.e. $d = 0$ on $\tilde{I}_d$. □

1.8. Given a 3-graded ideal $I = I_1 \oplus I_0 \oplus I_{-1}$ of a 3-graded Lie algebra L, it is easy to check by considering the canonical projections onto the subspaces I_i, $i = 0, \pm 1$, that

$$\operatorname{Der}(I,L) = \bigoplus_{i=0,\pm1,\pm2} (\operatorname{Der}(I,L))_i,$$

where each $(\operatorname{Der}(I,L))_i$ consists of derivations on I of degree i.

In particular, the derivation algebra $\operatorname{Der} L$ of a 3-graded Lie algebra L is 5-graded. From Proposition (1.7) and taking into account that for any derivation d on L, $\tilde{L}_d = L$, we get that $\operatorname{Der} L$ is in fact 3-graded when L is nondegenerate.

1.9. Corollary. *The derivation algebra of any nondegenerate 6-torsion free 3-graded Lie algebra is 3-graded itself.*

2. Lie algebras and Jordan pairs of $\mathfrak{M}$-quotients

2.1. A *filter* $\mathscr{F}$ on a Jordan system or a Lie algebra is a nonempty family of nonzero ideals such that

(F1) for any $I_1, I_2 \in \mathscr{F}$ there exists $I \in \mathscr{F}$ such that $I \subseteq I_1 \cap I_2$.

Moreover, $\mathscr{F}$ is a *power filter* if

(F2) for any $I \in \mathscr{F}$ there exists $K \in \mathscr{F}$ such that $K \subseteq I^\#$,

where $I^\# = [I,I]$ for Lie algebras, $I^\# = (Q_{I^+}I^-, Q_{I^-}I^+)$ for Jordan pairs, $I^\# = P_I I$ for Jordan triple systems, and $I^\# = I^2$ for Jordan algebras. We highlight the filter $\mathscr{F}_e$ of all essential ideals in V. When V is semiprime it is easy to show that $\mathscr{F}_e$ is a power filter, and that it coincides with the set $\mathscr{F}_s$ of all sturdy ideals in V (the proof of [32, Lemma 1.2] applies here with the obvious changes).

2.2. The notion of *Lie algebra of quotients* Q for a Lie algebra L is introduced in [32], and shown to be equivalent to Q being $\mathscr{F}_s$-absorbed into L [32, 2.15], i.e. for every nonzero element $q \in Q$ there exists a sturdy ideal I_q such that $0 \neq [q, I_q] \subseteq L$. This notion can be easily generalized to *Lie algebras of quotients with respect to a filter* just by requiring that the absorbing ideals belong to the filter.

In her work, Siles Molina builds the maximal Lie algebra of quotients of a semiprime Lie algebra as a direct limit of derivations on all the ideals of $\mathscr{F}_s$. Her construction can be generalized to the nonsemiprime case by considering power filters of sturdy ideals: The maximal Lie algebra of quotients $Q_{\mathscr{F}}(L)$ of L with respect to a power filter of sturdy ideals $\mathscr{F}$ can be then built as the direct limit of derivations on ideals of $\mathscr{F}$ (cf. [32, 3.1–3.4]),

$$Q_{\mathscr{F}}(L) = \varinjlim \operatorname{Der}(I, L), \qquad I \in \mathscr{F}.$$

Notice that powers of ideals in $\mathscr{F}$ are involved in the definition of a Lie product in $Q_{\mathscr{F}}(L)$ (cf. [32, 3.4]), and that is the reason why we require that $\mathscr{F}$ be a power filter. Moreover, as in [32, 3.6] the map $x \mapsto [\operatorname{ad} x, L]$ is a monomorphism of L into $Q_{\mathscr{F}}(L)$ (we need sturdiness of the ideals of $\mathscr{F}$ to get injectivity), and $Q_{\mathscr{F}}(L)$ is maximal among all Lie algebras of quotients of L with respect to $\mathscr{F}$: for any nonzero element s in a Lie algebra of quotients S of L with respect to $\mathscr{F}$ there exists $I_s \in \mathscr{F}$ such that $0 \neq [s, I_s] \subseteq L$, which implies (by sturdiness of the ideals in $\mathscr{F}$) that the map $s \mapsto [\operatorname{ad} s, I_s]$ is a monomorphism of S into $Q_{\mathscr{F}}(L)$.

2.3. Arguing as in [26, p. 160], when $\frac{1}{2} \in \Phi$ any power filter $\mathscr{F}$ on a 3-graded Lie algebra L is cofinal [26, 0.5] with the power filter $\tilde{\mathscr{F}} = \{\pi_1(I) \oplus \pi_0(I) \oplus \pi_{-1}(I) \mid I \in \mathscr{F}\}$, where π_j denote the projections onto L_j, $j = 0, \pm 1$. Therefore, $Q_{\mathscr{F}}(L) = Q_{\tilde{\mathscr{F}}}(L)$, i.e. when dealing with 3-graded Lie algebras we can assume that the filters we take consist only of 3-graded ideals.

2.4. Proposition. *For any nondegenerate 6-torsion free 3-graded Lie algebra and any power filter of sturdy ideals $\mathscr{F}$, the maximal Lie algebra of quotients $Q_{\mathscr{F}}(L)$ is 3-graded.*

Proof. As mentioned in (2.3), we can assume without loss of generality that every ideal in $\mathscr{F}$ is 3-graded.

Given an element $q \in Q_{\mathscr{F}}(L)$, we say q has degree i if there exists a graded ideal $I \in \mathscr{F}$ and a derivation $d \in \operatorname{Der}(I, L)$ of degree i in the equivalence class of q. Note that this definition does not depend on the graded ideal I and the derivation d chosen in q since we can always find a common domain for two derivations in q.

Now, by (1.8), $Q_{\mathscr{F}}(L)$ is 5-graded. Moreover, there are no elements of degree ± 2 in $Q_{\mathscr{F}}(L)$ because by (1.7) every derivation $d \in \operatorname{Der}(I, L)$ of degree ± 2 (note that I is nondegenerate by [36, Lemma 4]) can be restricted to an ideal of $\mathscr{F}$ contained in $[I, I]$ on which d is zero. $\square$

Following this notion for Lie algebras and the ideal absorption properties of Martindale systems of quotients in the associative setting (cf. [26, 1.3, 3.20]), we will introduce a similar notion for Jordan pairs.

2.5. Let V be a Jordan pair and consider a filter $\mathscr{F}$ on V. We say that a Jordan pair W is a *pair of Martindale-like quotients* (*pair of $\mathfrak{M}$-quotients, for short*) *of V with respect to $\mathscr{F}$ if W is $\mathscr{F}$*-absorbed into V, i.e. for each $0 \neq q \in W^\sigma$ there exists an ideal $I_q \in \mathscr{F}$ such that

$$\{q, I_q^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I_q^\sigma\} \subseteq V^\sigma$$

$$\{I_q^{-\sigma}, q, V^{-\sigma}\} \subseteq V^{-\sigma},$$

with $\{q, I_q^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I_q^\sigma\} \neq 0$ or $\{I_q^{-\sigma}, q, V^{-\sigma}\} \neq 0$.

2.6. Lemma. *Let W be a Jordan pair of $\mathfrak{M}$-quotients for a Jordan pair V with respect to a filter $\mathscr{F}$. Then $\{q, I^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I^\sigma\} + \{I^{-\sigma}, q, V^{-\sigma}\} \neq 0$ for any $0 \neq q \in W^\sigma$ and any sturdy ideal I in $\mathscr{F}$. Moreover, if $\mathscr{F}$ is a power filter, $\{q, I^{-\sigma}, I^\sigma\} + \{I^{-\sigma}, q, I^{-\sigma}\} \neq 0$.*

Proof. Let $0 \neq q \in W^\sigma$ be $\mathscr{F}$-absorbed into V by some $I_q \in \mathscr{F}$, and suppose that I is a sturdy ideal of V such that $\{q, I^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I^\sigma\} + \{I^{-\sigma}, q, V^{-\sigma}\} = 0$. Then, by [20, JP14], for any $a, b, u, v \in V$

$$\{v, u, \{q, a, b\}\} = \{q, a, \{v, u, b\}\} + \{\{v, u, q\}, a, b\} - \{q, \{u, v, a\}, b\},$$

so $\{v, u, \{q, a, b\}\} = 0$ whenever u or v belong to I. Similarly, $\{v, \{q, a, b\}, u\} = 0$ when u or v are in I using again JP14, so $\{q, a, b\} \in \mathrm{Ann}_V(I) = 0$ when the element $\{q, a, b\}$ is in V (for example, when a or b belong to I_q). Therefore, $\{q, I_q^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I_q^\sigma\} = 0$. Analogously, one can also show that $\{I_q^{-\sigma}, q, V^{-\sigma}\} = 0$, leading to a contradiction with q being $\mathscr{F}$-absorbed by I_q into V.

Moreover, if $\mathscr{F}$ is a power filter, we can consider an ideal K of $\mathscr{F}$ contained in $Q_I I$. Now $[K^+, V^-] + [V^+, K^-] \subseteq [\{I^+, I^-, I^+\}, V^-] + [V^+, \{I^-, I^+, I^-\}] \subseteq [I^+, I^-]$ using the Jacobi identity, hence

$$\{q, K^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, K^\sigma\} \subseteq \{q, I^{-\sigma}, I^\sigma\}$$

and by [20, JP16]

$$\{K^{-\sigma}, q, V^{-\sigma}\} \subseteq \{V^{-\sigma}, \{I^\sigma, I^{-\sigma}, q\}, I^{-\sigma}\} + \{V^{-\sigma}, I^\sigma, \{I^{-\sigma}, q, I^{-\sigma}\}\}.$$

In particular, $\{q, I^{-\sigma}, I^\sigma\} + \{I^{-\sigma}, q, I^{-\sigma}\} \neq 0$ since $\{q, K^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, K^\sigma\} + \{K^{-\sigma}, q, V^{-\sigma}\} \neq 0$. $\square$

As a consequence of this lemma, when dealing with filters of sturdy ideals one can find "common absorbing ideals" for any finite set of nonzero elements in a pair of $\mathfrak{M}$-quotients.

2.7. Corollary. *Let W be a Jordan pair of $\mathfrak{M}$-quotients for a Jordan pair V with respect to a filter of sturdy ideals $\mathscr{F}$. Then for every $0 \neq q_1, \ldots, q_n \in W$ there exists an ideal I in $\mathscr{F}$ that absorbs all the q_i, $i = 1, \ldots, n$.*

Proof. It is enough to consider an ideal $I \in \mathscr{F}$ contained in $\bigcap_{i=1}^{n} I_{q_i} \in \mathscr{F}$ and use (2.6) to show that it absorbs all the elements $q_1, \ldots, q_n$ into V in a nontrivial way. □

Now we show that regularity conditions such as semiprimeness, primeness and nondegeneracy are inherited by Jordan pairs of $\mathfrak{M}$-quotients.

2.8. Proposition. *Let $V \leqslant W$ be Jordan pairs such that W is a pair of $\mathfrak{M}$-quotients of V with respect to a filter $\mathscr{F}$. If V is nondegenerate* (*semiprime*, *prime*), *then W is nondegenerate* (*semiprime*, *prime*) *as well.*

Proof. It is straightforward to see that every nonzero ideal of W has nonzero intersection with V. Thus, if V is semiprime (prime) then so is W. Let $w \in W^{\sigma}$ be an absolute zero divisor of W, and let $y \in V^{-\sigma}$, $z \in V^{\sigma}$ be such that $\{w, y, z\} \in V^{\sigma}$. By [20, JP21], for any $a \in V^{-\sigma}$, we have

$$\begin{aligned} Q_{\{w,y,z\}}a &= Q_w Q_y Q_z a + Q_z Q_y Q_w a + D_{z,y} Q_w D_{y,z} a - Q_{w, Q_z Q_y w} a \\ &= -\{w, a, Q_z Q_y w\} \end{aligned}$$

since w is an absolute zero divisor. Let $b \in V^{-\sigma}$. By the above equality,

$$\begin{aligned} Q_{Q_{\{w,y,z\}}a} b = Q_{\{w,a,Q_z Q_y w\}} b &= -\{w, b, Q_{Q_z Q_y w} Q_a w\} \\ &= -\{w, b, Q_z Q_y Q_w Q_y Q_z Q_a w\} = 0, \end{aligned}$$

using again that w is an absolute zero divisor. Then, by the nondegeneracy of V, $Q_{\{w,y,z\}}a = 0$ for every $a \in V^{-\sigma}$, and

$$\{w, y, z\} = 0 \quad \text{whenever } \{w, y, z\} \in V^{\sigma}. \tag{8}$$

Let $w \in W^{\sigma}$ be a nonzero absolute zero divisor of W as before, and now let $x, z \in V^{-\sigma}$ be such that $\{x, w, z\} \in V^{-\sigma}$. By [20, JP20], for any $a \in V^{\sigma}$ we have

$$\begin{aligned} Q_{\{x,w,z\}}a &= Q_x Q_w Q_z a + Q_z Q_w Q_x a + Q_{x,z} Q_w Q_{x,z} a - \{Q_x w, a, Q_z w\} \\ &= \{Q_x w, a, Q_z w\}, \end{aligned}$$

since w is an absolute zero divisor. Hence, by [20, JP21], for $b \in V^{\sigma}$ we have

$$\begin{aligned} Q_{Q_{\{x,w,z\}}a} b = Q_{\{Q_x w, a, Q_z w\}} b &= Q_{Q_x w} Q_a Q_{Q_z w} b + Q_{Q_z w} Q_a Q_{Q_x w} b \\ &\quad -\{Q_z w, b, Q_{Q_x w} Q_a Q_z w\} + \{Q_x w, a, Q_{Q_z w}\{a, Q_x w, b\}\} = 0, \end{aligned}$$

since w is an absolute zero divisor, so is any $Q_x w$ [20, JP3]. Again, by nondegeneracy of V, we get

$$\{x, w, z\} = 0 \quad \text{whenever } \{x, w, z\} \in V^{-\sigma}. \tag{9}$$

Therefore, from (8) and (9) and taking into account that W is $\mathscr{F}$-absorbed into V, we get $w = 0$, which completes the proof. □

In the following proposition we show how the notions of Jordan pairs of $\mathfrak{M}$-quotients and Lie algebras of quotients are related. The connection of Jordan pairs with Lie algebras goes through the TKK construction. Before dealing with systems of quotients, let us relate filters on Jordan pairs and their TKK Lie algebras.

2.9. Lemma. *Let $\mathscr{F}$ be a filter on a Jordan pair V, and consider*

$$\mathscr{F}_{\mathrm{TKK}} = \{K = \mathrm{id}_{\mathrm{TKK}(V)}((K^+,K^-)) \mid (K^+,K^-) \in \mathscr{F}\},$$

where $\mathrm{id}_{\mathrm{TKK}(V)}((K^+,K^-)) = K^+ \oplus ([K^+,V^-]+[V^+,K^-]) \oplus K^-$ *is the ideal of* $\mathrm{TKK}(V)$ *generated by* $K^+ \cup K^-$. *Then,*

(i) $\mathscr{F}_{\mathrm{TKK}}$ *is a filter on* $\mathrm{TKK}(V)$, *and it is a power filter if* $\mathscr{F}$ *is so.*
(ii) *The ideal* K *generated in* $\mathrm{TKK}(V)$ *by an ideal* (K^+,K^-) *of* V *is sturdy in* $\mathrm{TKK}(V)$ *if and only if* (K^+,K^-) *is sturdy in* V.

Proof. (i) It is straightforward to check that the intersection of two ideals in $\mathscr{F}_{\mathrm{TKK}}$ contains again an ideal in the filter. Moreover, the Lie square of the ideal K generated in $\mathrm{TKK}(V)$ by (K^+,K^-) contains the ideal generated in $\mathrm{TKK}(V)$ by any ideal of V contained in $(Q_{K^+}K^-, Q_{K^-}K^+)$.

(ii) Let K be the ideal of $\mathrm{TKK}(V)$ generated by a sturdy ideal (K^+,K^-) of V. If $x = x^+ + x_0 + x^- \in \mathrm{Ann}_{\mathrm{TKK}(V)}(K)$, then both x^+ and x^- belong to $\mathrm{Ann}_V((K^+,K^-))$: $\{x^\sigma,K^{-\sigma},V^\sigma\} + \{x^\sigma,V^{-\sigma},K^\sigma\} \subseteq [x^\sigma,K_0]$ and the latter is zero because it is the σ-projection of $[x,K_0]=0$ onto V^σ, and similarly, $\{K^{-\sigma},x^\sigma,V^{-\sigma}\}$ is the $-\sigma$-projection of $[[K^{-\sigma},x],V^{-\sigma}]=0$ onto $V^{-\sigma}$. Hence $x=x_0$ and for any $y\in V^\sigma$, $[x,y]\in V^\sigma$. Using the Jacobi identity several times, we can show that $[x,y]\in \mathrm{Ann}_V((K^+,K^-))=0$, so $[x,y]=0$ for any $y\in V^\sigma$, $\sigma=\pm$, which by definition of $\mathrm{TKK}(V)$ gives $x=0$. The converse is straightforward. □

2.10. Theorem. *For any Jordan pair of* $\mathfrak{M}$*-quotients* W *of* V *with respect to a filter of sturdy ideals* $\mathscr{F}$, $\mathrm{TKK}(V) \subseteq \mathrm{TKK}(W)$. *Moreover, if we consider a power filter* $\mathscr{F}$ *of sturdy ideals on* V, $\mathrm{TKK}(W)$ *is a Lie algebra of quotients for* $\mathrm{TKK}(V)$ *with respect to the power filter of sturdy ideals* $\mathscr{F}_{\mathrm{TKK}}$.

Proof. Recall [31, 1.5(6)] that

$$\mathrm{TKK}(V) \cong (V^+ \oplus [V^+,V^-] \oplus V^-)/C_V,$$

$$\mathrm{TKK}(W) \cong (W^+ \oplus [W^+,W^-] \oplus W^-)/C_W,$$

where the Lie brackets are taken in any Lie algebra L containing V and W as subpairs, and $C_V=\{x\in[V^+,V^-] \mid [x,V^\sigma]=0,\ \sigma=\pm\}$, $C_W=\{x\in[W^+,W^-] \mid [x,W^\sigma]=0,\ \sigma=\pm\}$. To show that $\mathrm{TKK}(V)\subseteq \mathrm{TKK}(W)$ it is enough to prove that

$$C_W = \{q\in[W^+,W^-] \mid [q,V^+]=0=[q,V^-]\}.$$

Set $q\in[W^+,W^-]$ with $[q,V^+]=0=[q,V^-]$ and suppose that there exists $0\neq p\in W^\sigma$ such that $[q,p]\neq 0$. Since $0\neq p,[q,p]\in W^\sigma$ (by 2.7) there exist an ideal I in $\mathscr{F}$ that absorbs both p and $[q,p]$ into V in a nontrivial way. Then for all $v\in V^{-\sigma}$ and all $y\in I^{-\sigma}$, $0=[q,\{v,p,y\}]=[q,[[v,p],y]]=[[q,[v,p]],y]+[[v,p],[q,y]]=[[[q,v],p],y]+$

$[[v,[q,p]],y]=\{v,[q,p],y\}$, hence $\{V^{-\sigma},[q,p],I^{-\sigma}\}=0$. Similarly, $\{[q,p],V^{-\sigma},I^{\sigma}\}+\{[q,p],I^{-\sigma},V^{\sigma}\}=0$, giving a contradiction.

Now suppose that $\mathscr{F}$ is a power filter of sturdy ideals and let $0\neq q\in \mathrm{TKK}(W)$. Then there exist $q_i^+\in W^+$ and $q_i^-\in W^-$, $i=0,1,\ldots,n$, such that $q=q_0^+ +\sum_{i=1}^n [q_i^+,q_i^-]+q_0^-$. By (2.7) let I be a sturdy ideal in $\mathscr{F}$ which absorbs all the q_i^σ, $n=0,1,\ldots,n$, $\sigma=\pm$. Now let us consider an ideal $K\in\mathscr{F}$ contained in $Q_I I$ (which exists because $\mathscr{F}$ is a power filter). Let us denote by $\tilde{K}$ the ideal of $\mathrm{TKK}(V)$ generated by K,

$$\tilde{K}=K^+\oplus([K^+,V^-]+[V^+,K^-])\oplus K^-,$$

which belongs to $\mathscr{F}_{\mathrm{TKK}}$. We claim that $[x,[\tilde{K},\tilde{K}]]\subseteq \mathrm{TKK}(V)$, which is a consequence of the Jacobi identity and the following facts:

(1) $[q_i^\sigma,K^{-\sigma}]\subseteq[V^+,I^-]+[I^+,V^-],\quad i=0,\ldots,n,\ \sigma=\pm,$
(2) $[q_i^\sigma,([K^+,V^-]+[V^+,K^-])]\subseteq V^\sigma,\quad i=0,\ldots,n,\ \sigma=\pm,$
(3) $[[q_i^+,q_i^-],\tilde{K}]\subseteq V^+\oplus([q_i^+,V^-]+[V^+,q_i^-])\oplus V^-,\quad i=1,\ldots,n,$
(4) $[[[q_i^+,q_i^-],\tilde{K}],\tilde{K}]\subseteq\tilde{K}+\mathrm{TKK}(V),\quad i=1,\ldots,n.$

Indeed, (1) follows from $[q_i^\sigma,K^{-\sigma}]\subseteq[q_i^\sigma,\{I^{-\sigma},I^\sigma,I^{-\sigma}\}]\subseteq[\{q_i^\sigma,I^{-\sigma},I^\sigma\},I^{-\sigma}]+[\{I^{-\sigma},q_i^\sigma,I^{-\sigma}\},I^\sigma]+[I^{-\sigma},\{q_i^\sigma,I^{-\sigma},I^\sigma\}]\subseteq[V^+,I^-]+[I^+,V^-]$, (2) holds directly, and (3) and (4) are consequences of the previous containments.

By (2.9), $\mathscr{F}_{\mathrm{TKK}}$ is a power filter of sturdy ideals so there exists a sturdy ideal $M\in\mathscr{F}_{\mathrm{TKK}}$ which is contained in $[\tilde{K},\tilde{K}]$ and such that $[q,M]\subseteq\mathrm{TKK}(V)$. To show that $[q,M]\neq 0$, let us first suppose that $q_0^\sigma\neq 0$ for some $\sigma=\pm$. Then, by (2.6), either $0\neq\{q_0^\sigma,M^{-\sigma},V^\sigma\}+\{q_0^\sigma,V^{-\sigma},M^\sigma\}=[q,([M^{-\sigma},V^\sigma]+[V^{-\sigma},M^\sigma])]\subseteq[q,M]$ or $0\neq\{V^{-\sigma},q_0^\sigma,M^{-\sigma}\}=[V^{-\sigma},[q,M^{-\sigma}]]$, which gives again $[q,M]\neq 0$. If $0\neq q=\sum[q_i^+,q_i^-]\in\mathrm{TKK}(W)$, there exists $y\in W^\sigma$ for some $\sigma=\pm$ such that $[q,y]\neq 0$. We can then find an ideal $N\in\mathscr{F}_{\mathrm{TKK}}$ that absorbs both y and $[q,y]$ into $\mathrm{TKK}(V)$ and any $P\in\mathscr{F}_{\mathrm{TKK}}$ contained in $[M\cap N,M\cap N]$ satisfies $0\neq[[q,y],P]\subseteq[q,[y,P]]+[[q,P],y]\subseteq[q,[[y,N],M]]+[[q,M],y]\subseteq[q,M]+[[q,M],y]$ just by repeating the argument above for nonzero elements in $W^+\oplus W^-$.

Therefore, we have shown that any $0\neq q\in\mathrm{TKK}(W)$ can be absorbed into $\mathrm{TKK}(V)$ in a nontrivial way by ideals in $\mathscr{F}_{\mathrm{TKK}}$. $\square$

2.11. Example. In Zelmanov's classification of strongly prime Jordan pairs we find examples of $\mathfrak{M}$-quotients. Recall [2, 5.4] that a strongly prime hermitian Jordan pair V has the form

$$H(R,*)\lhd V\leqslant H(Q(R),*)$$

for a $*$-prime associative pair R and its associative Martindale pair of symmetric quotients $Q(R)$. We claim that $H(Q(R),*)$ is a Jordan pair of $\mathfrak{M}$-quotients of $H(R,*)$. For any $q\in H(Q(R),*)^\sigma$ there exists a $*$-ideal I of R that absorbs q into R in an associative way (cf. [26, 3.20; 9, 2.5(i)]), hence all the triple products $\{H(I,*)^\sigma,H(R,*)^{-\sigma},q\}$, $\{H(R,*)^\sigma,H(I,*)^{-\sigma},q\}$ and $\{H(I,*)^{-\sigma},q,H(R,*)^{-\sigma}\}$ belong to $H(R,*)$. Moreover, for any $0\neq q\in H(R,*)^\sigma$ and any nonzero $*$-ideal I of R, we have that either $\{q,H(I,*)^{-\sigma},H(I,*)^\sigma\}$ or $\{H(I,*)^{-\sigma},q,H(I,*)^{-\sigma}\}$ are nonzero. Indeed, we can consider the Jordan ideal $H(I,*)$, which is nonzero by [3, 2.1], and a nonzero ideal K of $H(R,*)$ contained in the Jordan cube $Q_{H(I,*)}H(I,*)$ of $H(I,*)$ (notice that the cube of $H(I,*)$ is a nonzero semi-ideal of $H(R,*)$ by semiprimeness of $H(R,*)$ and K can be taken as the intersection of the two ideals of $H(R,*)$ determined by the + and − parts of $Q_{H(I,*)}H(I,*)$). Then, by [9, 3.2] $0 \neq Q_q K^{-\sigma} = \{q, K^{-\sigma}, q\} \subseteq \{q, \{H(I,*)^{-\sigma}, H(I,*)^{\sigma}, H(I,*)^{-\sigma}\}, q\} \subseteq \{\{q, H(I,*)^{-\sigma}, H(I,*)^{\sigma}\}, H(I,*)^{-\sigma}, q\} + \{H(I,*)^{\sigma}, \{H(I,*)^{-\sigma}, q, H(I,*)^{-\sigma}\}, q\}$, so in particular either $\{q, H(I,*)^{-\sigma}, H(I,*)^{\sigma}\}$ or $\{H(I,*)^{-\sigma}, q, H(I,*)^{-\sigma}\}$ must be nonzero.

2.12. Example. In the literature of Jordan systems we find analogues of concepts that in the associative setting are tied to Martindale rings of symmetric quotients. In that sense, extended centroids and extended central closures were defined by Montaner (cf. [29]) in order to give Martindale and Posner type theorems for Jordan systems, and are also related with systems of $\mathfrak{M}$-quotients.

Given a nondegenerate Jordan pair V, it is trivial that any element $q = \sum [\lambda_i \otimes x_i]$ in the extended central closure $C(V)V$ of V can be absorbed into V by an essential ideal. Indeed, it suffices to consider the intersection I of all the domains I_i of the λ_i, which is again an essential ideal and absorbs the element q into V. Therefore, the extended central closure of a nondegenerate Jordan pair is a pair of $\mathfrak{M}$-quotients of the original Jordan pair with respect to the filter of all essential ideals.

2.13. Example. Jordan systems of $\mathfrak{M}$-quotients allow a unified study of Jordan structures, without the usual distinction of PI and non-PI cases. A combination of the two examples above is also described in terms of systems of $\mathfrak{M}$-quotients.

Let V be a vector space over a field Φ of dimension $\neq 2$, and let $q: V \to \Phi$ be a nondegenerate quadratic form on V with associated bilinear form $q(x,y) := q(x+y) - q(x) - q(y)$, for $x, y \in V$. Then $V_1 = (V, V)$ becomes a Jordan pair for the product given by $Q_x y = q(x,y)x - q(x)y$. It is called the Clifford pair defined by q, and it is simple [13, p. 14, ex. 4]. By (2.12), $C(V_1)V_1$ is a Jordan pair of $\mathfrak{M}$-quotients of V_1 with respect to the filter $\mathscr{F}_1 = \{V_1\}$. On the other hand, if we consider the set of symmetric elements $V_2 = H(R,*)$ of a Φ-associative pair R with involution $*$ which is $*$-prime, we know from (2.11) that $H(Q(R),*)$ is a Jordan pair of $\mathfrak{M}$-quotients of V_2 with respect to the filter $\mathscr{F}_2$ of all nonzero ideals of V_2. Then, the Jordan pair $C(V_1)V_1 \oplus H(Q(R),*)$ is a Jordan pair of $\mathfrak{M}$-quotients of $V_1 \oplus V_2$ with respect to the filter $\mathscr{F} = \{V_1 \oplus I \mid I \in \mathscr{F}_2\}$.

2.14. Example. Jordan systems of $\mathfrak{M}$-quotients include relevant scalar extensions. Indeed, if J is a strongly prime unital Jordan algebra over a domain Φ acting on J without torsion (for example, $\Phi = \Gamma(J)$, the centroid of J), then $V(J)$ is a strongly prime Jordan pair and $V(J) \otimes_\Phi \Phi^{-1}\Phi$ is a Jordan pair of $\mathfrak{M}$-quotients of $V(J)$ with respect to the filter of all nonzero ideals of $V(J)$.

3. The maximal Jordan pair of $\mathfrak{M}$-quotients

In this paragraph we build the maximal Jordan pair of $\mathfrak{M}$-quotients for any given nondegenerate Jordan pair V with respect to a power filter of sturdy ideals $\mathscr{F}$. Our construction is based on the fact that the maximal Lie algebra of quotients of a nondegenerate 3-graded Lie algebra is again 3-graded. From now on we will assume that $\frac{1}{6} \in \Phi$.

3.1. Construction of the maximal Jordan pair of $\mathfrak{M}$-quotients. If V is a nondegenerate Jordan pair, then $\mathrm{TKK}(V)$ is a nondegenerate 3-graded Lie algebra (by [9, 1.6]). Let us consider a power filter $\mathscr{F}$ of sturdy ideals on V and the power filter again of sturdy ideals $\mathscr{F}_{TKK}$, and let us build the maximal Lie algebra of quotients $Q_{\mathscr{F}_{\mathrm{TKK}}}(\mathrm{TKK}(V))$ with respect to $\mathscr{F}_{\mathrm{TKK}}$, which is 3-graded by (2.4). Notice that $(Q_{\mathscr{F}_{\mathrm{TKK}}}(\mathrm{TKK}(V))_1, Q_{\mathscr{F}_{\mathrm{TKK}}}(\mathrm{TKK}(V))_{-1})$ is a Jordan pair itself since $\frac{1}{6} \in \Phi$ (1.2), and it will be called *the maximal Jordan pair of $\mathfrak{M}$-quotients of V with respect to the filter $\mathscr{F}$*, and will be denoted by $Q_{\mathscr{F}}(V)$.

3.2. Theorem. *Given a nondegenerate Jordan pair V and a power filter of sturdy ideals $\mathscr{F}$, the map $v \mapsto [\mathrm{ad}\, v, \mathrm{TKK}(V)]$ defines a monomorphism of V into $Q_{\mathscr{F}}(V)$. Moreover, $Q_{\mathscr{F}}(V)$ is a pair of $\mathfrak{M}$-quotients of V with respect to $\mathscr{F}$, and it is the maximal Jordan pair satisfying this property.*

Proof. To show that V can be seen as a subpair of $Q_{\mathscr{F}}(V)$ it is enough to consider the composition of maps

$$V^\sigma \hookrightarrow \mathrm{TKK}(V) \hookrightarrow Q_{\mathscr{F}_{\mathrm{TKK}}}(\mathrm{TKK}(V)) \to Q_{\mathscr{F}}(V)^\sigma,$$

where the first map above is just the inclusion of V into its TKK Lie algebra, the second one is the imbedding through the adjoint map of the Lie algebra $\mathrm{TKK}(V)$ into its maximal Lie algebra of quotients with respect to $\mathscr{F}_{\mathrm{TKK}}$ [32, 3.6], and the third map is the projection of the 3-graded Lie algebra $Q(\mathrm{TKK}(V))$ onto its 1 and -1 parts.

By construction, every nonzero element q in $Q_{\mathscr{F}}(V)^\sigma$ $(\sigma = \pm)$ can be $\mathscr{F}_{\mathrm{TKK}}$-absorbed into $\mathrm{TKK}(V)$ by an ideal of the form $K^+ \oplus ([K^+, V^-] + [V^+, K^-]) \oplus K^-$, for $(K^+, K^-) \in \mathscr{F}$. Therefore, if $0 \neq [q, K^{-\sigma}] \in \mathrm{IDer}(V)$, then $0 \neq \{q, K^{-\sigma}, V^\sigma\} + \{K^{-\sigma}, q, V^{-\sigma}\} \subseteq V^+ \oplus V^-$, and, otherwise, $0 \neq [q, ([K^+, V^-] + [V^+, K^-])] = \{q, V^{-\sigma}, K^\sigma\} + \{q, K^{-\sigma}, V^\sigma\} \subseteq V^\sigma$, i.e. $Q_{\mathscr{F}}(V)$ is a Jordan pair of $\mathfrak{M}$-quotients of V with respect to $\mathscr{F}$.

Finally, maximality of $Q_{\mathscr{F}}(V)$ follows by (2.10) from the maximality of the Lie algebra $Q_{\mathscr{F}_{\mathrm{TKK}}}(\mathrm{TKK}(V))$ with respect to $\mathscr{F}_{TKK}$. Indeed, $Q_{\mathscr{F}}(V)$ coincides with the 1 and -1 parts of the maximal Lie algebra of quotients of $\mathrm{TKK}(V)$ with respect to $\mathscr{F}_{\mathrm{TKK}}$. □

4. Jordan triple systems of $\mathfrak{M}$-quotients

In this paragraph we introduce the notion of Jordan triple system of quotients with respect to a filter and study some of its properties. The connection of pairs and triple systems through the functor $V(\ )$ is the key point for our construction of a maximal triple system of $\mathfrak{M}$-quotients with respect to a fixed filter.

4.1. Let T be a Jordan triple system and let $\mathscr{F}$ be a filter on T. We say that the Jordan triple system Q is a *Jordan triple system of* $\mathfrak{M}$*-quotients of* T *with respect to* $\mathscr{F}$ if Q is $\mathscr{F}$-absorbed into T, i.e. for each $0 \neq q \in Q$ there exists an ideal $I_q \in \mathscr{F}$ such that

$$0 \neq \{q, I_q, T\} + \{q, T, I_q\} + \{I_q, q, T\} \subseteq T.$$

Notice that this notion is compatible with the definition of Jordan pair of $\mathfrak{M}$-quotients. Indeed, $Q \geqslant T$ is a Jordan triple system of $\mathfrak{M}$-quotients of T with respect to a filter $\mathscr{F}$ if and only if $V(Q)$ is a Jordan pair of $\mathfrak{M}$-quotients of $V(T)$ with respect to the filter $V(\mathscr{F}) = \{(I,I) \mid I \in \mathscr{F}\}$.

Furthermore, $\mathscr{F}$ is a power filter on a Jordan triple system T if and only if $V(\mathscr{F})$ is a power filter on $V(T)$, and any ideal I in T is sturdy if and only if (I,I) is a sturdy ideal of $V(T)$, hence $\mathscr{F}$ is a filter of sturdy ideals on T if and only if $V(\mathscr{F})$ is a filter of sturdy ideals on $V(T)$.

Therefore, going through the functor $V(\ )$, we can rephrase (2.6)–(2.8) in terms of Jordan triple systems.

4.2. Lemma. *Let Q be a Jordan triple system of $\mathfrak{M}$-quotients for a Jordan triple system T with respect to a filter $\mathscr{F}$. Then $\{q,I,T\} + \{q,T,I\} + \{I,q,T\} \neq 0$ for any $0 \neq q \in Q$ and any sturdy ideal I in $\mathscr{F}$. Moreover, if $\mathscr{F}$ is a power filter, $\{q,I,I\} + \{I,q,I\} \neq 0$.*

4.3. Corollary. *Let Q be a Jordan triple system of $\mathfrak{M}$-quotients for a Jordan triple system T with respect to a filter of sturdy ideals $\mathscr{F}$. Then for every $0 \neq q_1, \ldots, q_n \in Q$ there exists an ideal I in $\mathscr{F}$ that absorbs all the q_i, $i = 1, \ldots, n$.*

4.4. Proposition. *Let $T \leqslant Q$ be Jordan triple systems such that Q is a triple system of $\mathfrak{M}$-quotients of T with respect to a filter $\mathscr{F}$. If T is nondegenerate (semiprime, prime), then Q is nondegenerate (semiprime, prime) as well.*

Proof. For (semi)primeness, just notice that any nonzero ideal of Q has nonzero intersection with T. And for the inheritance of nondegeneracy, use (2.8) together with the fact that any Jordan triple system is nondegenerate if and only its associated double pair is nondegenerate. □

4.5. Construction of the maximal Jordan triple system of $\mathfrak{M}$-quotients. Let T be a nondegenerate Jordan triple system and $\mathscr{F}$ a power filter of sturdy ideals on T.

Let $V(T)$ be the double Jordan pair associated to T and $V(\mathscr{F})$ the power filter of sturdy ideals on $V(T)$ induced by $\mathscr{F}$. Recall that the maximal Jordan pair $Q_{V(\mathscr{F})}(V(T))$ of $\mathfrak{M}$-quotients of $V(T)$ arises as the 1 and -1 parts of the 3-graded Lie algebra

$$Q_{V(\mathscr{F})_{\mathrm{TKK}}}(\mathrm{TKK}(V(T))) = \{[d,I] \| I \in V(\mathscr{F})_{\mathrm{TKK}} \text{ and } d \in \mathrm{Der}(I, \mathrm{TKK}(V(T)))\}.$$

Let us prove that Jordan pair $Q_{V(\mathscr{F})}(V(T)) = (\mathscr{Q}^+, \mathscr{Q}^-)$ is isomorphic to the double pair of a triple system.

Let us denote by $\Psi : Q_{V(\mathscr{F})}(V(T)) \to Q_{V(\mathscr{F})}(V(T))^{\mathrm{op}}$ the map defined by $\Psi([d,I]) = [d^{\mathrm{ex}},I]$ (cf. (1.5)), which is clearly a Jordan pair isomorphism (the isomorphism Ψ can be seen to act as the exchange map on $V(T)$). Therefore, we can define a triple product in $\mathscr{Q}^+$ by $\{x,y,z\} = \{x,\Psi(y),z\}$. This triple system will be called *the maximal Jordan triple system of $\mathfrak{M}$-quotients of T with respect to $\mathscr{F}$*, and will be denoted by $Q_{\mathscr{F}}(T)$.

4.6. Theorem. *Given a nondegenerate Jordan triple system T and a power filter of sturdy ideals $\mathscr{F}$, T imbeds into $Q_{\mathscr{F}}(T)$. Moreover, $Q_{\mathscr{F}}(T)$ is a triple system of $\mathfrak{M}$-quotients of T with respect to $\mathscr{F}$, and it is the maximal Jordan triple system satisfying this property.*

Proof. Since Ψ is the exchange map on $V(T)$, it implies that T can be seen as a triple subsystem of $Q_{\mathscr{F}}(T)$. Moreover, $Q_{\mathscr{F}}(T)$ is a Jordan triple system of $\mathfrak{M}$-quotients of T with respect to $\mathscr{F}$ since, by construction, the double pair $V(Q_{\mathscr{F}}(T))$ is a Jordan pair of $\mathfrak{M}$-quotients of $V(T)$ with respect to $V(\mathscr{F})$ (see (4.1)).

Finally, given any Jordan triple system S of $\mathfrak{M}$-quotients of T, $V(S)$ imbeds into $Q_{V(\mathscr{F})}(V(T))$, and Ψ acts again as the exchange map on $V(S)$: for every $s \in S$, and any $x,y \in T$ such that $\{x,y,\Psi(s)\} \in T$ and $\{x,y,s\} \in T$ (resp. $\{x,\Psi(s),y\} \in T$ and $\{x,s,y\} \in T$) we have that $\{x,y,(\Psi(s)-s)\} = \{\Psi(x),\Psi(y),\Psi(s)\} - \{x,y,s\} = \Psi(\{x,y,s\}) - \{x,y,s\} = 0$ (resp. $\{x,(\Psi(s)-s),y\} = 0$), so $s = \Psi(s)$ since S is a Jordan triple system of $\mathfrak{M}$-quotients of T. Therefore, S can be seen as a triple subsystem of $Q_{\mathscr{F}}(T)$, hence $Q_{\mathscr{F}}(T)$ is maximal. □

5. Jordan algebras of $\mathfrak{M}$-quotients

5.1. Let J be a Jordan algebra and $\mathscr{F}$ a filter on J. We will say that a Jordan algebra Q is a *Jordan algebra of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$* if Q is $\mathscr{F}$-absorbed into J, i.e. for each $0 \neq q \in Q$ there exists an ideal $I_q \in \mathscr{F}$ such that

$$0 \neq q \circ I_q \subseteq J.$$

Let us relate the notions of algebra and triple system of $\mathfrak{M}$-quotients. Notice that every (power) filter $\mathscr{F}$ of ideals on a Jordan algebra J is also a (power) filter of ideals on the underlying triple system J_T.

5.2. Proposition. *Let J be a Jordan algebra and let $\mathscr{F}$ be a power filter of sturdy ideals on J. Then Q is a Jordan algebra of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$ if and only if Q_T is a Jordan triple system of $\mathfrak{M}$-quotients of J_T with respect to $\mathscr{F}$.*

Proof. First notice that J is sturdy because it contains some sturdy ideals. Therefore, by (0.4) $\mathscr{F}$ is a power filter of sturdy ideals on the underlying triple system J_T.

Now suppose that Q is a Jordan algebra of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$, and let q be a nonzero element of Q. If I_q is an ideal of $\mathscr{F}$ such that $0 \neq q \circ I_q \subseteq J$, let us see that any ideal K of $\mathscr{F}$ contained in $P_{I_q} I_q$ $\mathscr{F}$-absorbs q into J_T.

For every $x_1,x_2,x_3 \in I_q$ and every $z \in J$, by [20, JP16] and (0.1)(2),

$$\{q,\{x_1,x_2,x_3\},z\} = \{q,x_3,\{x_2,x_1,z\}\} + \{q,x_1,\{z,x_3,x_2\}\} - \{q,\{x_1,z,x_3\},x_2\} \subseteq \{q,I_q,I_q\} \subseteq J,$$

i.e. $\{q,K,J\} \subseteq J$, and similarly $\{q,J,K\} + \{J,q,K\} \subseteq J$.

Moreover, if we take $y \in I_q$ such that $q \circ y \neq 0$, then $\{q \circ y,K,J\} + \{q \circ y,J,K\} + \{J,q \circ y,K\} \neq 0$ since K is a sturdy ideal of J_T. Now it suffices to use the formula (which is [23, 0.7] linearized on the second variable)

$$t \circ \{x,y,z\} = \{t \circ x,y,z\} - \{x,t \circ y,z\} + \{x,y,t \circ z\} \tag{10}$$

to get that $\{q,J,K\} + \{J,q,K\} + \{q,K,J\} \neq 0$. Indeed, if $\{q \circ y,K,J\} \neq 0$, then $0 \neq \{y \circ q,K,J\} \subseteq y \circ \{q,K,J\} + \{q,y \circ K,J\} + \{q,K,y \circ J\} \subseteq y \circ \{q,K,J\} + \{q,K,J\} + \{q,K,J\}$, giving $\{q,J,K\} + \{J,q,K\} + \{q,K,J\} \neq 0$. Use a similar argument if $\{q \circ y,J,K\} \neq 0$ or if $\{J,q \circ y,K\} \neq 0$.

Conversely, if Q_T is a Jordan triple system of $\mathfrak{M}$-quotients of J_T with respect to $\mathscr{F}$ and q is a nonzero element of Q, there exists an ideal $I \in \mathscr{F}$ such that $0 \neq \{q,I,J\} + \{q,J,I\} + \{I,q,J\} \subseteq J$. If we consider an ideal $K \in \mathscr{F}$ contained in I^2, then

$$q \circ K \subseteq q \circ (I \circ I) \subseteq \{q,I,I\} \subseteq J$$

because $2q \circ (y_1 \circ y_2) = \{y_1,y_2,q\} + \{y_2,y_1,q\}$. Moreover, the product $q \circ K \neq 0$ since otherwise $\{q,K,K\} + \{K,q,K\} \subseteq (q \circ K) \circ K + q \circ K^2 = 0$, and that contradicts (4.2). □

5.3. Proposition. *Let Q be a Jordan algebra of $\mathfrak{M}$-quotients of a Jordan algebra J with respect to a power filter of sturdy ideals $\mathscr{F}$ on J. Then there exists a unital Jordan algebra $\tilde{Q}$, extension of Q, such that $\tilde{Q}$ is a Jordan algebra of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$.*

Proof. Let us denote by Q^1 the unital hull of Q and consider

$$K = \{q \in Q^1 \mid \exists I \in \mathscr{F} \text{ with } q \circ I = 0\}.$$

We claim that K is an ideal of Q^1: On the one hand, if $p,q \in K$ and we denote by I_p and I_q two ideals in $\mathscr{F}$ such that $p \circ I_p$ and $q \circ I_q$ are zero, then $(p+q) \circ (I_p \cap I_q) = 0$, hence there exists an ideal $I_{p,q} \in \mathscr{F}$ contained in $I_p \cap I_q$ such that $(p+q) \circ I_{p,q} = 0$, i.e. $p+q \in K$. On the other hand, for any $q \in K$ and $x = p + \lambda 1 \in Q^1$,

$$(q \circ (p+\lambda 1)) \circ t = (q \circ p) \circ t + 2\lambda(q \circ t) = (q \circ t) \circ p - (p \circ t) \circ q + 2\{q,p,t\} + 2\lambda(q \circ t) \tag{11}$$

for any $t \in Q$ by (0.1)(2). Let I_q be an ideal in $\mathscr{F}$ such that $q \circ I_q = 0$ and let $I' \in \mathscr{F}$ be an absorbing ideal for p (i.e. $p \circ I' \subseteq J$). Then for any $t_1,t_2,t_3 \in I_q \cap I'$ by (5.2)(10) we have

$$p \circ \{t_1,t_2,t_3\} = \{p \circ t_1,t_2,t_3\} - \{t_1,p \circ t_2,t_3\} + \{t_1,t_2,p \circ t_3\} \in I_q, \tag{12}$$

hence

$$(p \circ \{t_1,t_2,t_3\}) \circ q \in I_q \circ q = 0. \tag{13}$$

Moreover, if each $t_1, t_2, t_3 \in (I_q \cap I')^3$, by [20, JP16],

$$\begin{aligned}\{q, p, \{t_1, t_2, t_3\}\} &= \{q, \{t_2, t_1, p\}, t_3\} - \{q, t_2, \{t_3, p, t_1\}\} \\ &\quad + \{q, \{t_2, t_3, p\}, t_1\} \subseteq \{q, I_q, I_q\} \text{ (by (0.1)(2) and (12))} \\ &= 0. \end{aligned} \tag{14}$$

Therefore, if we take an ideal $I_{q,x}$ of $\mathscr{F}$ contained in $((I_q \cap I')^3)^3$ and use (11), (13) and (14) we get $(q \circ x) \circ I_{q,x} = 0$, i.e. $q \circ x \in K$.

Moreover, since $\mathscr{F}$ is a power filter of sturdy ideals on J, $K \cap Q = 0$: If $0 \neq x \in K \cap Q$, then there exists an absorbing ideal $I' \in \mathscr{F}$ such that $0 \neq x \circ I' \subseteq J \cap K$ (since K is an ideal of Q^1). Since $x \circ I'$ is contained into K, for any $y \in I'$ there exists $I_{x,y} \in \mathscr{F}$ such that $(x \circ y) \circ I_{x,y} = 0$, hence $x \circ y \in \mathrm{Ann}_J(I^2_{x,y})$ by [7, 2.5, 2.9], but $\mathrm{Ann}_J(I^2_{x,y}) = 0$ because $I^2_{x,y}$ contains a sturdy ideal of the filter, thus $x \circ I' = 0$, leading to a contradiction.

Finally, the quotient $\tilde{Q} = Q^1/K$ is a Jordan algebra of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$: Let $\overline{(p + \lambda 1)}$ be a nonzero element of $\tilde{Q}$ and consider an ideal I of $\mathscr{F}$ such that $I \circ p \subseteq J$. Then, by construction of $\tilde{Q}$, $I \circ \overline{(p + \lambda 1)}$ is nonzero and it is clearly contained in J. □

5.4. Construction of the maximal Jordan algebra of $\mathfrak{M}$-quotients. Let J be a nondegenerate Jordan algebra, let $\mathscr{F}$ be a power filter of sturdy ideals on J, and consider $Q_{\mathscr{F}}(J_T)$ the maximal Jordan triple system of $\mathfrak{M}$-quotients of J_T with respect to $\mathscr{F}$ (4.6). Since there exists a unital Jordan algebra $\tilde{J}$ of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$ by (5.3), $\tilde{J}_T \subseteq Q_{\mathscr{F}}(J_T)$ by (5.2) (notice that J is a Jordan algebra of $\mathfrak{M}$-quotients of itself).

Let us check that $Q_{\mathscr{F}}(J_T)$ has a Jordan algebra structure. If we denote the unit of $\tilde{J}$ by 1, we claim that $P_1 q = q$ for every $q \in Q_{\mathscr{F}}(J_T)$. Indeed, if $P_1 q - q \neq 0$ we could always find $x, y \in J$ such that $0 \neq \{(P_1 q - q), x, y\} + \{x, (P_1 q - q), y\} \in J$ and at the same time $\{q, x, y\} + \{x, P_1 q, y\} \in J$ using (4.3) because $Q_{\mathscr{F}}(J_T)$ is a Jordan triple system of $\mathfrak{M}$-quotients of J_T. But using that 1 is the unit of $\tilde{J}$ and [20, JP3], $\{P_1 q, x, y\} = \{P_1 q, x, P_1 y\} = P_1\{q, P_1 x, y\} = \{q, x, y\}$ and $\{x, P_1 q, y\} = P_1\{x, P_1 q, y\} = \{P_1 x, q, P_1 y\} = \{x, q, y\}$, leading to a contradiction.

Therefore, by (0.2) $Q_{\mathscr{F}}(J_T)$ is a unital Jordan algebra with product $p \circ q = \{p, 1, q\}$. This Jordan algebra will be called *the maximal Jordan algebra of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$ and denoted by $Q_{\mathscr{F}}(J)$.*

5.5. Theorem. *Given a nondegenerate Jordan algebra J and a power filter of sturdy ideals $\mathscr{F}$, J imbeds into $Q_{\mathscr{F}}(J)$. Moreover, $Q_{\mathscr{F}}(J)$ is a unital Jordan algebra of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$, and it is the maximal Jordan algebra satisfying this property.*

Proof. It is clear that J is a subalgebra of $Q_{\mathscr{F}}(J)$. Moreover, since the underlying triple system of $Q_{\mathscr{F}}(J)$ is a Jordan triple system of $\mathfrak{M}$-quotients of J_T, $Q_{\mathscr{F}}(J)$ is a Jordan algebra of $\mathfrak{M}$-quotients of J with respect to $\mathscr{F}$ by (5.2).

Furthermore, if S is a Jordan algebra of $\mathfrak{M}$-quotients of J, we can consider by (5.3) a unital Jordan algebra $\tilde{S}$ of $\mathfrak{M}$-quotients of J such that S is a subalgebra of $\tilde{S}$. So its underlying triple system $\tilde{S}_T$ is a Jordan triple system of $\mathfrak{M}$-quotients of J_T by (5.2) and $S \subseteq \tilde{S}_T \subseteq Q_{\mathscr{F}}(J) = Q_{\mathscr{F}}(J_T)$. Now, arguing as in (5.4), the unit $\tilde{1}_{\tilde{S}}$ of $\tilde{S}$ satisfies that $x \circ_{\tilde{S}} y = \{x, \tilde{1}_{\tilde{S}}, y\}$ and $x \circ_{\tilde{S}} y$ coincides with the algebra product $x \circ_Q y$ in $Q_{\mathscr{F}}(J)$. Finally, S is a subalgebra of $\tilde{S}$ and $\tilde{S}$ is a subalgebra of $Q_{\mathscr{F}}(J)$, so S is a subalgebra of $Q_{\mathscr{F}}(J)$, i.e. $Q_{\mathscr{F}}(J)$ is maximal among all Jordan algebras of $\mathfrak{M}$-quotients of J. $\square$

5.6. Final remark. In a forthcoming work it will be shown that Zelmanov's classification of strongly prime Jordan algebras [35] can be expressed in terms of algebras of $\mathfrak{M}$-quotients, unifying both PI and non-PI cases (see (2.11)–(2.13)).

Acknowledgements

The authors would like to thank Prof. J.A. Anquela for his careful reading of this manuscript and his valuable suggestions and comments, and the referee for his/her interesting remarks and suggestions.

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